International Journal of Theoretical Physics, Vol. 38, No. 9, 1999

D-Dimensional *q*-Harmonic Oscillator and *d*-Dimension *q*-Hydrogen Atom

Su Ka-Lin[1] **and Liu An-ling**[2]

Received May 1, 1999

A q-analogue of the D-dimensional harmonic oscillator is presented. A new realization of the quantum algebra $SU_q(1,1)$ · via the D-dimensional q-harmonic oscillator is found. A model of the d-dimensional q-hydrogen atom is constructed by means of the D-dimensional q-harmonic oscillator. The dimension D of the q-harmonic oscillator and the dimension d of the q-hydrogen atom are arbitrary.

In recent years, the q-analogue of the one-dimensional harmonic oscillator has been studied by several authors [1–3]. Realizations of the quantum algebra $SU_q(1,1)$ via the one-dimensional q-harmonic oscillator were suggested by Chaichian *et al.* [4, 5]. A model of the three-dimensional q-hydrogen atom has been constructed by Kibler and Negadi [6] and others [7, 8]. In this paper, we will present the q-analogue of the D-dimensional harmonic oscillator, find a new realization of the quantum algebra $SU_q(1,1)$ via the D-dimensional q-harmonic oscillator, and construct a model of the D-dimensional q-hydrogen atom by means of the D-dimensional q-harmonic oscillator. The dimension D of the q-harmonic oscillator and the dimension d of the q-hydrogen atom are arbitrary.

1. THE *q*-ANALOGUE OF THE *D*-DIMENSIONAL HARMONIC OSCILLATOR

The annihilation and creation operators of the D-dimensional q-harmonic oscillator $a_{q\alpha}$ and $a^+_{q\alpha}$ ($\alpha = 1, 2, 3, \ldots, D$) can be defined as

[1] Department of Physics, Yueyang Normal College, Yueyang 414000, China.
[2] Department of Physics, Changsha Electric Power College, Changsha 410000, China.

0020-7748/99/0900-228 9$16.00/0

$$a_{q\alpha} = \left\{ \frac{[N_\alpha + 1]}{N_\alpha + 1} \right\}^{1/2} a_\alpha, \qquad a_{q\alpha}^{+} = a_\alpha^{+} \left\{ \frac{[N_\alpha + 1]}{N_\alpha + 1} \right\}^{1/2} \tag{1}$$

where a_α and a_α^+ are the annihilation and creation operators of the D-dimensional harmonic oscillator, $N_\alpha = a_\alpha^+ a_\alpha$, and $[x] = (q^x - q^{-x})/(q - q^{-1})$. These operators satisfy the following relations:

$$a_{q\alpha}^{+} a_{q\alpha} = [N_\alpha], \qquad a_{q\alpha} a_{q\alpha}^{+} = [N_\alpha + 1], \qquad a_{q\alpha} a_{q\alpha}^{+} - q a_{q\alpha}^{+} a_{q\alpha} = q^{-N_\alpha} \tag{2}$$

$$[a_{q\alpha}, a_{q\beta}] = [a_{q\alpha}^{+}, a_{q\beta}^{+}] = [N_\alpha, N_\beta] = 0 \tag{3}$$

where $\alpha = 1, 2, 3, \ldots, D$ and $\beta = 1, 2, 3, \ldots, D$. The Hamiltonian H_q' of the D-dimensional q-harmonic oscillator can be defined as

$$H_q' = \frac{1}{2} \sum_{\alpha=1}^{D} (a_{q\alpha}^{+} a_{q\alpha} + a_{q\alpha} a_{q\alpha}^{+}) = \frac{1}{2} \sum_{\alpha=1}^{D} ([N_\alpha] + [N_\alpha + 1]) \tag{4}$$

where we have assumed $\hbar\omega = 1$; the eigenequation of H_q' is

$$H_q' \big| n_1, n_2, \ldots, n_{\alpha_1}, \ldots, n_D \rangle = E_{qN}' \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle \tag{5}$$

where E_{qN}' is the energy eigenvalue of H_q', and $\big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle$ is the corresponding eigenvector. Because

$$N_\alpha \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle = n_\alpha \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle \tag{6}$$

we have

$$[N_\alpha] \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle = [n_\alpha] \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle \tag{7}$$

Equation (5) becomes

$$H_q' \big| n_1, ni2, \ldots, n_\alpha, \ldots, n_D \rangle = \left\{ \frac{1}{2} \sum_{\alpha=1}^{D} ([n_\alpha] + [n_\alpha + 1] \right\} \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle \tag{8}$$

Then we have

$$E_{qN}' = \frac{1}{2} \sum_{\alpha=1}^{D} ([n_\alpha] + [n_\alpha + 1]) \tag{9}$$

Equation (9) is the energy spectrum formula of the D-dimensional q-harmonic oscillator.

2. THE D-DIMENSIONAL q-HARMONIC OSCILLATOR REALIZATION OF THE QUANTUM ALGEBRA $SU_q(1,1)$

We use the annihilation and creation operators $a_{q\alpha}$ and $a_{q\alpha}^{+}$ ($\alpha = 1, 2, 3, \ldots, D$) of the D-dimensional q-harmonic oscillator to construct the operators K_1', K_2', and K_3'.

$$K_1' = \frac{1}{4} \sum_{\alpha=1}^{D} [(a_{q\alpha}^{+})_2 + a_{q\alpha}^2]$$

$$K_2' = -\frac{i}{4} \sum_{\alpha=1}^{D} [(a_{q\alpha}^{+})_2 - a_{q\alpha}^2] \tag{10}$$

$$K_3' = \frac{1}{4} \sum_{\alpha=1}^{D} (a_{q\alpha}^{+}\, a_{q\alpha} + a_{q\alpha}\, a_{q\alpha}^{+})$$

Then we define K_+' and K_-' by $K_\pm' = K_1' \pm iK_2'$; it is easy to show that the operators K_+', K_-', and K_3' satisfy the following commutation relations:

$$[K_3', K_\pm'] = \pm K'_\pm, \qquad [K_+', K_-'] = -[2K_3'] \tag{11}$$

These are the familiar commutation relations of the quantum algebra SU_q (1,1). This indicates that we have realized the quantum algebra $SU_q(1,1)$ via the D-dimensional q-harmonic oscillator.

3. MODEL OF THE d-DIMENSIONAL q-HYDROGEN ATOM

In order to construct a model of the d-dimensional q-hydrogen atom, first we find the relationship between the d-dimensional hydrogen atom and the D-dimensional harmonic oscillator [9, 10]. We use $x_1, x_2, x_3, \ldots, x_d$ to express the d-dimensional hydrogen atom coordinate space and construct the operators K_1, K_2, and K_3 for the space [11]:

$$K_1 = \frac{1}{2}(r\Delta + r), \qquad K_2 = i\left[\frac{d-1}{2} + \sum_{j=1}^{d} x_j \frac{\partial}{\partial x_j}\right], \qquad K_3 = -\frac{1}{2}(r\Delta - r) \tag{12}$$

where $\Delta = \Sigma_{j=1}^{d}\, \partial^2/\partial x_j \partial x_j$ and $r = (\Sigma_{j=1}^{d}\, x_j^2)^{1/2}$. It is easy to show that the operators satisfy the following commutation relations:

$$[K_1, K_2] = -iK_3, \qquad [K_2, K_3] = iK_1, \qquad [K_3, K_1] = iK_2 \tag{13}$$

These relations show that the operators K_1, K_2, and K_3 constitute the $SU(1,1)$ algebra. We write the Hamilitonian of the d-dimensional hydrogen atom as

$$H = -\frac{1}{2}\Delta - \frac{1}{r} \tag{14}$$

where we have assumed $\hbar = \mu = e = 1$. Using Eq. (12), we can reduce Eq. (14) to

$$(K_1 + K_3)H = -\frac{1}{2}(K_1 - K_3) - 1 \tag{15}$$

The eigenequation of the Hanmiltonian H and eigenvalue can be written as [8, 9, 12, 13]

$$H|d, n\rangle = E_n|d, n\rangle \tag{16}$$

$$E_n = -\frac{1}{2}\frac{1}{[n + (d-3)/2]^2} \tag{17}$$

where E_n is the energy eigenvalue of the Hamiltonian H, and $|d, n\rangle$ is the corresponding eigenvector. From Eqs. (15) and (16), we obtain

$$\left\{-\left(\frac{1}{2} + E_n\right)K_1 + \left(\frac{1}{2} - E_n\right)K_3 - 1\right\}|d, n\rangle = 0 \tag{18}$$

Defining the function θ_n by

$$\cosh\theta_n = \frac{1 - 2E_n}{\sqrt{-8E_n}}, \qquad \sinh\theta_n = -\frac{1 + 2E_n}{\sqrt{-8E_n}} \tag{19}$$

and using the relation satisfied by the elements of the $SU(1,1)$ algebra

$$e^{-iK_2\theta_n}K_3e^{iK_2\theta_2} = K_3\cosh\theta_n + K_1\sinh\theta_n \tag{20}$$

we can be rewrite Eq, (18), as

$$\left\{K_3 - \frac{1}{\sqrt{-2E_n}}\right\}e^{-iK_2\theta_2}|d, n\rangle = 0 \tag{21}$$

This is an eigenequation of the operator K_3. Thus, we have transformed the eigenequation of the Hamiltonian of the d-dimensional hydrogen atom into an eigenequation of the operator K_3.

We use the annihilation and creation operators a_α and a_α^+ ($\alpha = 1, 2, 3 \ldots, D$) of the D-dimensional harmonic oscillator to realize the $SU(1,1)$ algebra given in ref. 14. The results are

$$K_1 = \frac{1}{4}\sum_{\alpha=1}^{D}[(a_\alpha^+)^2 + a_\alpha^2]$$

$$K_2 = -\frac{i}{4}\sum_{\alpha=1}^{D}[(a_\alpha^+)^2 - a_\alpha^2] \tag{22}$$

$$K_3 = \frac{1}{4} \sum_{\alpha=1}^{D} (a_\alpha^+ a_\alpha + a_\alpha a_\alpha^+)$$

obviously K_1, K_2, and K_3 satisfy Eq. (13). In other words, K_1, K_2, and K_3 constitute the $SU(1,1)$ algebra. The Hamiltonian H' of the D-dimensional harmonic osccillator is

$$H' = \frac{1}{2} \sum_{\alpha=1}^{D} (a_\alpha^+ a_\alpha + a_\alpha a_\alpha^+) \tag{23}$$

where we have assumed $\omega\hbar = 1$. The eigenequation and the eigenvalue of the operator H' can be written as

$$H' \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle = E'_N \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle \tag{24}$$

$$E'_N = N + D/2 \tag{25}$$

where E'_N is the energy eigenvalue of the operator H', $\big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle$ is the corresponding eigenvector, and N is the eigenvalue of the operator $\Sigma_{\alpha=1}^{D} a_\alpha^+$, $N = n_1 + n_2 + \cdots + n_D$. Comparing Eq. (22) with Eq. (23), we can rewrite Eq. (24) as

$$K_3 \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle = \frac{1}{2} E'_N \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle \tag{26}$$

This also is an eigenequation of the operator K_3. Comparing Eq. (21) with Eq. (26), one can find relations

$$\big| d, n \rangle = e^{-iK_2\theta_n} \big| n_1, n_2, \ldots, n_\alpha, \ldots, n_D \rangle \tag{27}$$

$$E_N = -\frac{2}{(E'_N)^2} \tag{28}$$

$$D = \frac{D}{2} + 1, \qquad n = \frac{n}{2} + 1 \tag{29}$$

where

$$K_2 = -\frac{i}{4} \sum_{\alpha=1}^{D} [(a_\alpha^+)^2 - a_\alpha^2]$$

Equations (27)–(29) are relate the d-dimensional hydrogen atom and the D-dimensional harmonic oscillator. From Eqs. (27)–(29) and (9), we find the energy spectrum of the d-dimensional q-hydrogen atom

$$E_{qn,n_1n_2\cdots n_D} = -\frac{2}{(E'_{qN})^2} = -\frac{8}{\{\Sigma_{\alpha=1}^{D}([n_\alpha] + [n_\alpha + 1])\}^2} \tag{30}$$

where $D = 2(d - 1)$, $N = 2(n - 1) = n_1 + n_2 + \cdots + n_D$, and the corresponding eigenvector is

$$\left|d, n\right\rangle_{q,n_1n_2\cdots n_D} = e^{-iK_2'\theta_{qn}}\left|n_1, n_2, \ldots, n_\alpha, \ldots, n_D\right\rangle \tag{31}$$

where

$$K_2' = -\frac{i}{4}\sum_{\alpha=1}^{D}[(a_{q\alpha}^{+})^2 - a_{q\alpha}^2]$$

The function θ_{qn} is defined by

$$\cosh\theta_{qn} = \frac{1 - 2E_{qn}}{\sqrt{-8E_{qn}}}, \qquad \sinh\theta_{qn} = -\frac{1 - 2E_{qn}}{\sqrt{-8E_{qn}}} \tag{32}$$

The d-dimensional q-hydrogen atom energy spectrum (30) shows that the d-dimensional q-hydrogen atom has the same ground energy level as the ordinary d-dimensional hydrogen atom; the excited energy levels ($n > 2$) of the d-dimensional q-hydrogen atom are split. For example, when $n = 2$, $d = 3$, we have

$$E_{q2,2000} = -\frac{8}{([2] + [3] + 3)^2}, \qquad E_{q2,1100} = -\frac{2}{([2] + 2)^2} \tag{33}$$

and when $n = 3$, $d = 3$, we have

$$E_{q3,1111} = -\frac{1}{2([2] + 1)^2}, \qquad E_{q3,2200} = -\frac{2}{([2] + [3] + 1)^2}$$

$$E_{q3,3100} = -\frac{8}{([3] + [4] + [2] + 3)^2}, \qquad E_{q3,2110} = -\frac{8}{(3[2] + [3] + 3)^2}$$

$$E_{q3,4000} = -\frac{1}{([4] + [5] + 3)^2} \tag{34}$$

Equations (30) and (31) show that we have constructed a model of the d-dimensional q-hydrogen atom; when $d = 3$, Eq. (30) is just Gora's result [7] when $q \to 1$, the results (30) and (31) become the classical case [see (27) and (28)]. Therefore, our work is consistent.

REFERENCES

1. Le-Man Kuang, *J. Phys. A* **26**, L1079 (1993).
2. A. S Zhedanov, *J. Phys. A* **25**, L713 (1992).

3. A. J. Macfarlane, *J. Phys. A* **22**, 4581 (1989).
4. M. Chaichian *et al.*, *Phys. Rev. Lett.* **65**, 980 (1990).
5. J. Katriel and A. I. Solomon, *J. Phys. A* **24**, 2093 (1991).
6. M. Kibler and T. Negadi, *J. Phys. A* **24**, 5283 (1991).
7. J. Gora, *J. Phys. A* **25**, L1281 (1992).
8. Song Xing-Chang and Liao Li, *J. Phys. A* **25**, 623 (1992).
9. Zeng Gao-Jian, Su Ka-lin, and Li Min, *Phys. Rev. A* **50**, 4373 (1994).
10. Zeng Gao-Jian, Su Ka-Lin, and Li Min, *Chin. Phys. Lett.* **11**, 724 (1994).
11. Su Ka-Lin, *Chin. Phys. Lett.* **14**, 721 (1997).
12. A. O. Barut and I. H. Duru, *Proc. R. Soc. A* **333**, 217 (1973).
13. M. M. Nieto, *Am. J. Phys.* **47**, 1067 (1979).
14. T. Lisowki, *J. Phys. A* **25**, 1259, (1992).